Assuming that for each dollar received by the son, the father will receive z dollars, and noting that the money received will vary inversely as the number of hours required to saw and split 1 cord of wood, we have

$$z = \frac{k+1}{kx} \Big/ \frac{k+2}{2kx} = 1 + \frac{k}{k+2}.$$

From which it is evident that the money received by the father depends directly upon k. Since $\lim_{k \doteq 0} z = 1$ and $\lim_{k \doteq \infty} z = 2$, then for $k > 0$, we have $1 < z < 2$.

Also solved by C. E. BARDSLEY, H. C. BRADLEY, H. N. CARLETON, MICHAEL GOLDBERG, and DANIEL KRETH.

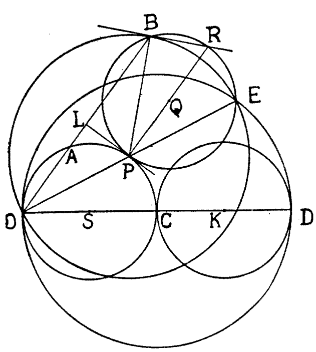
2861 [1920, 428]. Proposed by B. F. FINKEL, Drury College.

Obtain by plane geometry, *i.e.*, without the use of calculus, a construction for finding points on the envelope of a system of circles whose diameters are chords of a fixed circle passing through a given point on it. Also determine geometrically the nature of the locus.

I. SOLUTION BY AUGUSTUS BOGARD, College of St. Theresa, Winona, Minn.

The problem of finding the envelope of a system of curves is that of finding a curve that is at every point tangent to some curve of the system. That is, at any point on the envelope the envelope and some curve of the system have a common tangent, and they lie on the same side of it.

Let O be the fixed point, C the center of the given circle, and OCD the diameter from O, and let OE be any other chord from O, making with OD the angle θ. OE is then a diameter and P, its mid-point, is the center of one of the circles of the given system. At P, on the further side of OE, construct an angle $EPB = 2\theta$, the side of the angle cutting the circle OE at B. Then B is a point of the envelope required.



Proof. Draw the circles having OC and CD as diameters, and the circle PEB. Let Q be the center and PQR the diameter from P of PEB. PQR bisects the angle EPB and this circle is tangent at P to the circle OC. Further, chord $PB =$ chord PO and so diameter $PR =$ diameter $OC =$ diameter CD. Hence the circle whose center is at Q may be thought of as rolling without slipping on the circle OC, and P is the center of the curvature at B of the curve described by this point. It follows that the line BR is tangent to the curve at B. It is also tangent at B to the circle OE, the circle of the given system, passing through B. Hence B is a point on the envelope. This analysis shows that the curve is described in the same manner as the cardioid, and is therefore a cardioid. Its polar equation is easily derived as follows: Considering O as the pole, OD as the initial line, the angle $DOB = \phi$ as the vectorial angle of B, and OB its radius vector, we have $OB = OE \cos\theta = OD \cos^2\theta = a(\cos 2\theta + 1) = a(1 + \cos\phi)$, where $a = OC$. That is, $\rho = a(1 + \cos\phi)$, which is the equation of a cardioid.

II. SOLUTION BY OTTO DUNKEL, Washington University.

Using the notation in A. Bogard's solution, the middle point P of the chord OE of the fixed circle, of radius OC, lies on the circle OC. It may be shown easily that the tangent at P to the circle OC is parallel to the tangent to the given circle at E. Hence the construction given for the point B in the MONTHLY (*1921*, 182) is equivalent to the following: Draw the tangent at E to the given fixed circle, then the point B is the foot of the perpendicular from O to the tangent. Let A be the point in which OB cuts the circle OC, then $AB = CE = a$. Hence the point B is found by the usual method for constructing a cardioid, *i.e.*, by drawing chords OA to the circle OC and prolonging each a length $AB = OC = a$.

III. Solution, Historical Notes, and Remarks by R. C. Archibald, Brown University.

In an article by T. de St-Laurent, *Annales de Mathématiques Pures et Appliquées*, July, 1826, vol. 17, page 15, it is found that the cardioid is the catacaustic of a reflecting circle with respect to a luminous point on its circumference. In a footnote to this result Gergonne remarked that the catacaustic is the envelope of the space traversed by a moving circle of variable radius, constantly having its centre on the circumference of a circle concentric with the reflecting circle [and of one third the radius], and passing through a fixed point of this circumference.

But the result is only a very special case of a well-known theorem by Quetelet:[1] The envelope of the circles passing through a fixed point, and whose centers lie on a given curve, C, is the pedal with respect to the fixed point of a curve similar to C but of double its linear dimensions.

Colin Maclaurin, in a memoir[2] written in 1718, at twenty years of age, was not only the first to show that the cardioid is the first positive pedal of a circle, with respect to a point on its circumference, but also the first to find the equations of the positive and negative pedals of the cardioid with respect to its cusp.

That the envelope of the circles in B. F. Finkel's problem is a cardioid was shown by B. Price who gave an analytic proof[3] in 1852. A geometrical proof of this result is readily derived by inversion[4] of the theorem: If the vertex of a right angle move along a straight line and one side pass through a fixed point O, the envelope of the other side will be a parabola with focus at O.

Conversely, if a series of circles through the cusp of a cardioid are tangent to it, their cuspidal diameters will be chords of the cardioid's axial circle (Stubbs,[5] 1843).

It is interesting to note the relation of Quetelet's theorem to Otto Dunkel's solution above, and to his paper "The relation of caustics to certain envelopes" (1921, 182–183).

Also solved by Joseph Rosenbaum, A. V. Richardson, and F. L. Wilmer.

2864 [1920, 482]. Proposed by C. B. HALDEMAN, Ross, O.

If S is a side of a regular undecagon inscribed in a circle of radius unity, show that

$$S^5 - (S^4 - 3S^2 - 1)\sqrt{11} - 11S = 0.$$

Solution by A. M. Harding, University of Arkansas.

It is evident that $S = 2 \sin \pi/11$. Hence our problem is to find an equation one of whose roots is $2 \sin \pi/11$. The complex eleventh roots of unity, which occur in conjugate pairs, viz.

$$\cos \frac{2k\pi}{11} \pm i \sin \frac{2k\pi}{11}, \qquad k = 1, 2, 3, 4, 5.$$

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[1] *Nouveaux Mémoires de l'Académie de Bruxelles*, vol. 3, 1826, p. 91 (memoir presented February 3, 1823); the theorem is stated as follows: La caustique par refléxion pour une courbe quelconque éclairée par un point brillant, est la developpée d'une autre courbe, laquelle a la propriété d'être l'envelope de tous les cercles qui ont leurs centres sur la courbe réfléchissante, et qui passent par le point brillant." Quetelet showed also that the "autre courbe" was a "secondary caustic," the locus determined by dropping a perpendicular, OL, from the luminous point, O, on the tangent to the reflecting curve, and producing this perpendicular an equal distance to B.

[2] C. Maclaurin, "Tractatus de curvarum constructione & mensura ubi plurimae series curvarum infinitae vel rectis mensurantur vel ad simpliciores curvas reducunter," *Philosophical Transactions of the Royal Society* of London, vol. 30, pp. 803–812; abridgment, vol. 6, 1809, pp. 356–362. The geometrical construction for points on the cardioid and its pedals is touched upon by Maclaurin in his *Treatise of Fluxions*, 1742, vol. 2.

[3] B. Price, *Infinitesimal Calculus*, vol. 1, 1852, p. 417. It was also proposed and solved in *Educational Times*, June and October, 1853, and June, 1856. There are many other discussions of the problem.

[4] C. Taylor, *An Introduction to the Ancient and Modern Geometry of Conics*, Cambridge, 1881, p. 357.

[5] J. W. Stubbs, "On the application of a new geometric method to the geometry of curves and surfaces," *Philosophical Magazine*, vol. 23, pp. 338–347.